Solving Dirichlet and Poisson Problems on Graphs by means of Equilibrium Measures

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Equilibrium Measure and Green Function

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Abstract

We aim here at obtaining an explicit expression of the solution of the Dirichlet and Poisson problems on graphs. To this end, we consider the Laplacian of a graph as a kernel on the vertex set, V, in the framework of Potential Theory. Then, the properties of such a kernel allow us to obtain for each proper vertex subset the equilibrium measure that solves the so-called *equilibrium problem*. As a consequence, the Green function of the Dirichlet problems, the generalized Green function of the Poisson problems and the solution of the condenser principle are obtained solely in terms of equilibrium measures for suitable subsets. In particular, we get a formula for the effective resistance between any pair of vertices of a graph. Specifically, $r_{xy} = \frac{1}{n}(\nu_x(y) + \nu_y(x))$, where ν_z denotes the equilibrium measure for the set $V - \{z\}$. In any case, the equilibrium measure for a proper subset is accomplished by solving a Linear Programming Problem.

1 Introduction

It is well known that a flow of electric current in a network has a random walk as probabilistic counterpart. This follows from the fact that both can be seen as the solution of a suitable Dirichlet problem. When the network is finite and each edge has resistance 1, the network is modeled by a finite graph, and then the above mentioned problem is certainly a Dirichlet problem with respect to the discrete Laplacian of the graph, [8, 4].

A classical Dirichlet problem is a boundary value problem with respect to the Laplacian operator. Its solution could be expressed as a Dirichlet potential with respect to the Green kernel, [2, 11]. The main difficulty in this framework is to obtain an explicit expression of such a solution, since the Green kernel is a power series.

In the context of Graph Theory, the discrete Laplacian is no more than a matrix. Hence, it can be considered as a kernel on the vertex set and therefore it is possible to build a Potential Theory with respect to such a kernel. This point of view will take validity as the Laplacian kernel enjoys properties that lead to obtain explicit expressions of the solution of the Dirichlet problems. With hindsight, we will show that the Laplacian kernel verifies fundamental principles that allow solving the so-called *equilibrium problem* for any proper vertex subset.

By solving equilibrium problems for suitable subsets, we will build the Green function associated with each Dirichlet problem and hence its solution. This technique is also valid to obtain the solution, when it exists, of the Poisson problem, that is, the "degenerate" Dirichlet problem in which the boundary is empty. We also prove that the solution of the Dirichlet problem with piecewise constant boundary condition 0-1 solves the so-called *condenser principle*. The particular case where the boundary consist of two vertices allows us to get a formula for the effective resistance between any pair of vertices of a graph.

To summarize, the results here obtained for the Dirichlet and Poisson problems mainly hinge on the knowledge of the equilibrium measures that are solutions of appropriate equilibrium problems. Let us point out that the properties of the Laplacian, considered as a kernel in the framework of the Potential Theory, allow us to accomplish the effective computation of such measures by solving Linear Programming Problems.

Throughout the paper, $\Gamma = (V, E)$ denotes a (simple and finite) connected graph, with vertex set V, $|V| = n$, and edge set E, $|E| = m$. The distance from x to y is denoted by $d(x, y)$ and $d = d(\Gamma) = max\{d(x, y) : x, y \in V(\Gamma)\}$ stands for the *diameter* of Γ. The set of vertices adjacent to x is denoted by $\Gamma(x)$ and its cardinality is the *degree* of x, $\delta(x) = |\Gamma(x)|$. Given $F \subset V$, we denote by F^c its complement in V, and we consider the subsets $\delta(F) = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$ and $\mathrm{Ext}(F) = F^c \backslash \delta(F)$. In addition, for $x \in F$, we call *exterior degree of x with respect to F* the number $\partial^-(x) = |\Gamma(x) \cap F^c|$.

The *Laplacian matrix* of Γ is the $n \times n$-matrix $\mathcal{L} = \mathcal{L}(\Gamma)$ indexed by the vertices of Γ,

whose entries $\mathcal{L}_{xy}$ are given by $\mathcal{L}_{xy} = -1$ if x is adjacent to y, in short $x \sim y$, $\mathcal{L}_{xx} = \delta(x)$ and $\mathcal{L}_{xy} = 0$ otherwise. The matrix $\mathcal{L}$ is symmetric and positive semidefinite. Finally, $\mathcal{L}_{_F}$ will denote the $(|F|\times|F|)$-matrix associated with the restriction of $\mathcal{L}$ to the set F.

2 The Equilibrium Problem

In this section we present some results from Potential Theory that have been published by the authors in [1]. However we include them by completeness and because they will be used later. With this aim, we consider as underlying space the vertex set V of a graph Γ and its Laplacian matrix $\mathcal{L}$ as a kernel on V. We take advantage from this new point of view because the Laplacian of a graph, considered as a kernel, enjoys general properties which can easily be applied to solve the Dirichlet and Poisson Problems.

The sets $\mathcal{M}(V)$ and $\mathcal{M}^+(V)$ of measures and positive measures on V are identified with $\mathbb{R}^n$ and the positive cone of $\mathbb{R}^n$, respectively. So, if $\mu \in \mathcal{M}(V)$, then its *support* and its *mass* are given by $S(\mu) = \{x \in V : \mu(x) \neq 0\}$ and $||\mu|| = \sum\limits_{x\in V} |\mu(x)|$. For each vertex $x \in V$, ε_x stands for the Dirac measure on x whereas the measure $\sum\limits_{x\in V} \varepsilon_x$ will be denoted by $\mathbf{1}$. In addition, we denote by $\mathcal{M}^1(V)$ the set of positive measures on V with unit mass and if $F \subset V$, $\mathcal{M}(F) = \{\mu \in \mathcal{M}(V) : S(\mu) \subset F\}$.

If $\mu \in \mathcal{M}(V)$ the *potential of* μ is given by

$$\mathcal{L}\mu(x) = \sum_{y\sim x}(\mu(x) - \mu(y)) = \delta(x)\mu(x) - \sum_{y\sim x}\mu(y),$$

and the *energy of* μ is the value

$$I(\mu) = \langle \mathcal{L}\mu, \mu\rangle = \sum_{(x,y)\in E}(\mu(x) - \mu(y))^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n$.

Let us start by showing that the Laplacian kernel verifies the energy and maximum principles, which will be the key to solve the *equilibrium problem* for a proper subset F of V:

$$\text{Find } \nu \in \mathcal{M}^+(F) \text{ such that } \mathcal{L}\nu(x) = 1 \text{ if } x \in F.$$

In fact, we will prove that this problem has a unique solution $\nu^F \in \mathcal{M}^+(F)$ called the *equilibrium measure for* F.

Proposition 2.1 *The Laplacian kernel verifies the energy principle, i.e., $\mathcal{L}$ is strictly positive definite on $\{\mu \in \mathcal{M}(V) : \sum\limits_{x\in V} \mu(x) = 0\}$.*

Proof. It is clear that $I(\mu) \geq 0$ for all $\mu \in \mathcal{M}(V)$. Moreover, $I(\mu) = 0$ iff $\mu(x) = \mu(y)$ when $(x, y) \in E$. Hence, $I(\mu) = 0$ iff $\mu = a\mathbf{1}$, $a \in \mathbb{R}$, since Γ is connected. ∎

Let us point out that the energy principle is well-known (see for instance [5]). Moreover, it is equivalent to the fact that I is strictly convex on $\mathcal{M}^1(V)$, that is, $I(\mu - \nu) > 0$ for all $\mu, \nu \in \mathcal{M}^1(V)$ with $\mu \neq \nu$. To see this equivalence, it suffices to observe that

$$\Big\{\mu \in \mathcal{M}(V) : \sum_{x \in V} \mu(x) = 0\Big\} = \Big\{t(\mu - \nu) : t \geq 0 \ \text{ and } \ \mu, \nu \in \mathcal{M}^1(V)\Big\}.$$

Proposition 2.2 *The Laplacian kernel verifies the maximum principle, i.e.,*

$$\max_{x \in V} \mathcal{L}\mu(x) = \max_{x \in S(\mu)} \mathcal{L}\mu(x) \quad \textit{for all} \ \ \mu \in \mathcal{M}^+(V).$$

Proof. Given $\mu \in \mathcal{M}^+(V)$ and $F = S(\mu)$, it is verified that $\mathcal{L}\mu(x) = -\sum_{y \sim x} \mu(y) \leq 0$ if $x \in \delta(F)$ and $\mathcal{L}\mu(x) = 0$ if $x \in \mathrm{Ext}(F)$. Therefore it suffices to prove that there exists a vertex $x \in F$ such that $\mathcal{L}\mu(x) \geq 0$. Let $x \in F$ such that $\mu(x) = \max_{y \in F} \mu(y)$. Then $\mathcal{L}\mu(x) \geq 0$. ∎

Next we tackle the solution of the equilibrium problem. With this aim, we first prove the existence of measures whose potentials are constants.

Proposition 2.3 *For each $F \subset V$ there exists a unique $\sigma \in \mathcal{M}^1(F)$ whose potential is constant on F.*

Proof. Suppose that $\sigma \in \mathcal{M}^1(F)$ is such that $\mathcal{L}\sigma(x) = a$, $a \in \mathbb{R}$, for all $x \in F$. Then $\langle \mathcal{L}\sigma, \sigma \rangle = a$ and hence $a = I(\sigma)$. In particular this implies that $a \geq 0$.

Now, we prove that $\sigma \in \mathcal{M}^1(F)$ verifies $\mathcal{L}\sigma(x) = I(\sigma)$ for all $x \in F$ if and only if $\mathcal{L}\sigma(x) \geq I(\sigma)$ for all $x \in F$. Clearly, it is enough to prove the only if condition. Suppose that $\mathcal{L}\sigma(x) - I(\sigma) \geq 0$ for all $x \in F$, then

$$0 \leq \sum_{x \in S(\sigma)} (\mathcal{L}\sigma(x) - I(\sigma))\sigma(x) = \langle \mathcal{L}\sigma, \sigma \rangle - I(\sigma) = 0,$$

which implies that $\mathcal{L}\sigma(x) = I(\sigma)$ for all $x \in S(\sigma)$. Therefore $\mathcal{L}\sigma(x) \leq I(\sigma)$ for all $x \in V$, because $\mathcal{L}$ verifies the maximum principle and hence, $\mathcal{L}\sigma(x) = I(\sigma)$ for all $x \in F$.

On the other hand, $\mathcal{L}\sigma(x) \geq I(\sigma)$ for all $x \in F$ is equivalent to $\langle \mathcal{L}\sigma, \mu - \sigma \rangle \geq 0$ for all $\mu \in \mathcal{M}^1(F)$. But, this last condition is the Euler inequality relative to the minimization problem

$$\min_{\mu \in \mathcal{M}^1(F)} I(\mu).$$

So, as I is convex on $\mathcal{M}^1(F)$, σ verifies the Euler inequality iff I attains its minimum value on $\mathcal{M}^1(F)$ at σ. Furthermore, the extremal measure is unique since I is strictly convex on $\mathcal{M}^1(F)$. Finally, there exists an extremal measure because $\mathcal{M}^1(F)$ is compact and I is continuous. ∎

Corollary 2.4 *For each F a proper subset of V there exists a unique equilibrium measure for F, ν^F. Moreover, $S(\nu^F) = F$.*

Proof. If σ is the measure given in Proposition 2.3, then $I(\sigma) > 0$ and it suffices to consider $\nu^F = I(\sigma)^{-1}\sigma$.

On the other hand, suppose that there exists $x \in F$ such that $\nu^F(x) = 0$. Then, $\mathcal{L}\nu^F(x) \leq 0$ contradicting that $\mathcal{L}\nu^F(x) = 1$. ∎

We now prove a monotoniticity property for the equilibrium measures. Roughly speaking, the mass of the equilibrium measure at each vertex decreases when the vertex subset decreases.

Proposition 2.5 *If $H \subset F$ are proper subsets of V, then $\nu^F \geq \nu^H$.*

Proof. Let us denote $\mu = \nu^F - \nu^H$. It suffices to prove that if $x \in F$ is such that $\mu(x) = \min_{z \in F} \mu(z)$, then $\mu(x) \geq 0$. Note that $\mathcal{L}\mu \geq 0$ in F and hence

$$0 \leq \mathcal{L}\mu(x) = \delta(x)\mu(x) - \sum_{\substack{z \sim x \\ z \in F}} \mu(z) \leq \partial^-(x)\mu(x).$$

If $\partial^-(x) > 0$, then $\mu(x) \geq 0$. If $\partial^-(x) = 0$, then $0 \leq \sum_{z \sim x} (\mu(x) - \mu(z)) \leq 0$. Therefore $\mu(z) = \mu(x)$ for all $z \sim x$. Repeating this reasoning for a vertex $z \sim x$ and using the fact that Γ is connected and F is a proper set, there would exist a vertex $w \in F$ such that $\mu(w) = \mu(x)$ and $\partial^-(w) > 0$. ∎

The proof of the above proposition shows in fact, that $\mathcal{L}_F$ is a monotone matrix, that is, for each $\mu \in \mathcal{M}(F)$ such that $\mathcal{L}_F\mu \geq 0$ it is verified that $\mu \geq 0$, (see [7].)

We will finish this section by giving two alternative ways to obtain the equilibrium measure for F, or equivalently the unique measure σ of the Proposition 2.3. The first one is based on the energy minimization. Specifically, for each F, σ can be obtained as the

solution of a Convex Quadratic Programming Problem, namely

$$\begin{array}{c} \min \\ \mu \geq 0 \\ \sum \mu(x) = 1 \end{array} \quad \sum_{x,\, y \,\in\, F} \mathcal{L}(x,y)\mu(x)\mu(y).$$

The second method enables us to obtain the pair $(\sigma, I(\sigma))$ as the solution of a Linear Programming Problem. For see this, we first prove that σ is also the unique solution of a suitable extremal problem relative to the potential.

Proposition 2.6 *For each $F \subset V$, the problem*

$$\min_{\mu \in \mathcal{M}^1(F)} \max_{x \in F} \mathcal{L}\mu(x)$$

has as sole solution the unique measure $\sigma \in \mathcal{M}^1(F)$ whose potential is constant on F.

Proof. From the proof of Proposition 2.3, $\mathcal{L}\sigma(x) = I(\sigma)$ for $x \in F$. Then,

$$I(\sigma) = \max_{x \in F} \mathcal{L}\sigma(x) \geq \min_{\mu \in \mathcal{M}^1(F)} \max_{x \in F} \mathcal{L}\mu(x).$$

Conversely, let $\mu \in \mathcal{M}^1(F)$ and consider $a = \max_{x \in F} \mathcal{L}\mu(x)$. Then, $I(\mu) = \langle \mathcal{L}\mu, \mu \rangle \leq a$ which implies that $I(\sigma) \leq a$ and, *a fortiori*, $I(\sigma) \leq \min_{\mu \in \mathcal{M}^1(F)} \max_{x \in F} \mathcal{L}\mu(x)$. ∎

Adding a new variable that majorizes the potential values, the above min-max problem can be re-written as a minimization problem in the following way:

$$\begin{array}{c} \min \quad a \\ \mu \geq 0 \\ \sum \mu(x) = 1 \\ \mathcal{L}_{F}\mu \leq a\mathbf{1} \end{array}$$

Clearly, this is a Linear Programming Problem whose unique solution is the couple $(\sigma, I(\sigma))$.

3 Dirichlet and Poisson Problems

In this section we will tackle the discrete version of two classical problems, namely Dirichlet and Poisson Problems, in the context of graphs. For this, we bear in mind the natural identification between functions and measures on a finite space. Specifically, if $\mathcal{C}(V)$

denotes the set of all real functions on V, then $\mathcal{C}(V) \equiv \mathcal{M}(V)$. This identification will be used fruitfully in the rest of our work.

Given F a non-empty subset of V, $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(F^c)$, the *Dirichlet Problem* (see [2, p. 220]) consists in finding $u \in \mathcal{C}(V)$ such that

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } F, \\ u &= g \quad \text{in } F^c. \end{aligned} \tag{1}$$

The special case $F = V$, which was considered in [4], will be called here the *Poisson Problem* for V, (see [10].) Therefore, the Poisson Problem consists in finding $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}u = f \ \text{ in } \ V. \tag{2}$$

Our main goal is to obtain explicitly the solutions of both problems. To this end, we show that the so-called Green functions can be obtained by solving either $|F|+1$ or n equilibrium problems at most. Therefore, the Green functions could be obtained by solving Linear Programming Problems.

Let us start solving the Dirichlet Problem (1). Our methodology follows the standard steps both in the discrete and in the continuous settings (see [6] for a reference in the graph context).

Observe that this problem has at most a solution, since the homogeneous problem has as unique solution $u = 0$. As usual, the first step in the resolution of (1) is to transform it into a semi-homogeneous problem. Specifically, u is a solution of (1) iff $v = u - \hat{g}$ satisfies

$$\begin{aligned} \mathcal{L}v &= h \quad \text{in } F, \\ v &= 0 \quad \text{in } F^c, \end{aligned} \tag{3}$$

where $h = f - \mathcal{L}\hat{g}$ and $\hat{g} \in \mathcal{C}(V)$ is given by $\hat{g}(x) = g(x)$ if $x \in F^c$ and $\hat{g}(x) = 0$ if $x \in F$.

A function $G : V \times F \longrightarrow \mathbb{R}$ is called the *Green function* for F if $G_y(\cdot) = G(\cdot, y)$ is the solution of the semi-homogeneous Dirichlet Problem for $h = \varepsilon_y$, when $y \in F$, that is,

$$\begin{aligned} \mathcal{L}G_y &= \varepsilon_y \quad \text{in } F, \\ G_y &= 0 \quad \text{in } F^c. \end{aligned}$$

Clearly, the Green function for F is unique and the solution of (3) is given by

$$v(x) = \sum_{y \in F} G(x, y)h(y).$$

Next, we use the results of the preceding section to obtain an explicit formula of the Green function.

Proposition 3.1 *Let F be a proper subset of V and ν^F its equilibrium measure. If for each $y \in F$, ν_y^F denotes the equilibrium measure for $F - \{y\}$, the Green function for F is given by*

$$G(x,y) = \frac{\nu^F(y)}{||\nu^F|| - ||\nu_y^F||}\Big(\nu^F(x) - \nu_y^F(x)\Big).$$

Proof. Let $y \in F$ and consider $G_y = \dfrac{\nu^F(y)}{||\nu^F|| - ||\nu_y^F||}\Big(\nu^F - \nu_y^F\Big)$. Then, $G_y = 0$ in F^c, since $S(\nu^F), S(\nu_y^F) \subset F$. Moreover, as $\mathcal{L}\nu^F = \mathbf{1}$ in F and $\mathcal{L}\nu_y^F = \mathbf{1}$ in $F - \{y\}$ then $\mathcal{L}(\nu^F - \nu_y^F) = (1 - \mathcal{L}\nu_y^F(y))\varepsilon_y$ in F. On the other hand,

$$||\nu_y^F|| = \langle \mathcal{L}\nu^F, \nu_y^F\rangle = \langle \nu^F, \mathcal{L}\nu_y^F\rangle = ||\nu^F|| - \nu^F(y)\Big(1 - \mathcal{L}\nu_y^F(y)\Big).$$

Therefore $\mathcal{L}G_y = \varepsilon_y$ in F, and hence $G(x,y), x \in V, y \in F$ must be the Green function for F.

It is well known that the Green Function of the Laplacian has important properties. In particular, G is symmetric on $F \times F$ because $\mathcal{L}$ is symmetric. Furthermore, from the monotonicity property given in Proposition 2.5 and from the obtained expression for G, it follows that the Green Function is non-negative.

Now let us consider the Poisson Problem. To begin with, let us recall that the Poisson Problem has solution iff f verifies $\langle f, \mathbf{1}\rangle = 0$; i.e., for a fixed $z \in V$, $f = \sum\limits_{y \in V} f(y)(\varepsilon_y - \varepsilon_z)$. Moreover, if u is a solution of (2), then $u + a\mathbf{1}, a \in \mathbb{R}$ is also a solution (see [4]). Consequently, the equilibrium problem for V has no solution.

For a fixed vertex $z \in V$ a function $G^z : V \times V \longrightarrow \mathbb{R}$ will be called *z-Green Function* if for each $y \in V$, G_y^z is a solution of the following Poisson Problem,

$$\mathcal{L}G_y^z = \varepsilon_y - \varepsilon_z \ \text{ in } \ V$$

and moreover $G_y^z(z) = 0$. Clearly, the z-Green Function is unique and

$$u(x) = \sum_{y \in V} G^z(x,y) f(y),$$

is the unique solution of (2) that verifies $u(z) = 0$.

By using once again the equilibrium measure techniques, we get the expression of the z-Green Function.

Proposition 3.2 *If for each $y \in V$, ν_y denotes the equilibrium measure for $V - \{y\}$, the z-Green Function is given by*

$$G^z(x,y) = \frac{1}{n}(\nu_z(x) + \nu_y(z) - \nu_y(x)).$$

Proof. Clearly, $G^z(z,y)=0$ for all $y\in V$. Moreover, for each $y\in V$, $\langle \mathcal{L}\nu_y, \mathbf{1}\rangle = 0$, that is, $n-1+\mathcal{L}\nu_y(y)=0$. Hence, $\mathcal{L}\nu_y = \mathbf{1}-n\varepsilon_y$ and $\mathcal{L}(\nu_z+\nu_y(z)\mathbf{1}-\nu_y) = \mathcal{L}(\nu_z-\nu_y) = n(\varepsilon_y-\varepsilon_z)$. ∎

To end this section we show that the verification of a well known result in electrostatics, namely the condenser principle, is equivalent to the resolution of certain Dirichlet problem.

It is said that the *condenser principle* is satisfied if for any $E,H\subset V$, $E\cap H=\emptyset$, $E,H\neq\emptyset$, there exists $u\in\mathcal{C}(V)$ verifying:

$$\begin{aligned} \mathcal{L}u(x)=0, \quad 0\le u(x)\le 1 \qquad & \text{if } x\in (E\cup H)^c,\\ \mathcal{L}u(x)\ge 0, \qquad u(x)=1 \qquad & \text{if } x\in E,\\ \mathcal{L}u(x)\le 0, \qquad u(x)=0 \qquad & \text{if } x\in H. \end{aligned} \tag{4}$$

Proposition 3.3 *Let $E,H\subset V$, $E\cap H=\emptyset$, $E,H\neq\emptyset$. Then, u is the solution of (4) iff u is the solution of the following Dirichlet Problem*

$$\begin{aligned} \mathcal{L}u(x)=0 \qquad & \textit{if} \quad x\in (E\cup H)^c,\\ u(x)=1 \qquad & \textit{if} \quad x\in E,\\ u(x)=0 \qquad & \textit{if} \quad x\in H. \end{aligned} \tag{5}$$

Moreover,

$$u=\sum_{x\in E}\frac{\nu^{\{x\}\cup F}-\nu^F}{\nu^{\{x\}\cup F}(x)},$$

where $F=(E\cup H)^c$.

Proof. Firstly, we show that the above function is the solution of (5). Note that $S(u)\subset F\cup E$. Therefore $u=0$ in H. Besides, $S(\nu^F)=F$ and $S(\nu^{\{x\}\cup F})=\{x\}\cup F$, and hence from Corollary 2.4, $u=\mathbf{1}$ in E. Finally, $\mathcal{L}u=0$ in F, since $\mathcal{L}\nu^{\{x\}\cup F}=\mathbf{1}$ and $\mathcal{L}\nu^F=\mathbf{1}$ in F.

To conclude it suffices to prove that if u is the solution of (5), then u is the solution of (4). From Proposition 2.5, $\nu^{\{x\}\cup F}\ge\nu^F$ and hence $u\ge 0$. On the other hand, if $y\in H$, $\mathcal{L}u(y)=-\sum\limits_{z\sim y}u(z)\le 0$. Consider, $v=\mathbf{1}-u$, then v is the solution of

$$\begin{aligned} \mathcal{L}v(x)=0 \qquad & \text{if} \quad x\in F,\\ v(x)=0 \qquad & \text{if} \quad x\in E,\\ v(x)=1 \qquad & \text{if} \quad x\in H. \end{aligned}$$

Therefore, reasoning as above, $v\ge 0$, $\mathcal{L}v(y)\le 0$ if $y\in E$ and *a fortiori* $u\le\mathbf{1}$ and $\mathcal{L}u(y)\ge 0$ if $y\in E$. ∎

If we allow that $H = \emptyset$, then the condenser principle is known in the literature as *the equilibrium principle for E relative to the Laplacian operator* (see [2]) and it must not be mistaken for the equilibrium with respect to the Laplacian kernel we have used here. Note that unlike the continuous case, the equilibrium problem for E with respect to the Laplacian operator is not outstanding, since its solution is obviously $u = \mathbf{1}$.

4 Applications

There exists a variety of Dirichlet and Poisson Problems whose translation to the context of networks and random walks has a wide range of applications. In this section we deal with problem (5) in the special case $E = \{x\}$ and $H = \{y\}$. So, we consider Γ as an electrical network in which each edge has unit resistance. For concepts and results not given here we refer the reader to P. Doyle & J. Snell [8] and N. Biggs [3, 4].

One of the main problems in Network Theory is to calculate the effective resistance between any pair of vertices. If $x, y \in V$, the *effective resistance* between x and y is defined as $r_{xy} = u(x) - u(y)$, where $u \in \mathcal{C}(V)$ is any solution of the Poisson Problem $\mathcal{L}u = \varepsilon_x - \varepsilon_y$. Note that r_{xy} does not depend on the chosen solution and $r_{xy} = r_{yx}$. Clearly, from the definition of the y-Green Function we can take $u(z) = G^y(z, x)$ and hence $r_{xy} = G^y(x, x)$. Finally, from Proposition 3.2 we get

$$r_{xy} = \frac{1}{n}(\nu_x(y) + \nu_y(x)). \tag{6}$$

Using this expression we can also get formulas for the *effective conductance* between x and y, which is defined as $\gamma_{xy} = r_{xy}^{-1}$, or the *escape probability* denoted by $P_{\text{esc}}(x, y)$. This parameter is the probability that a walk starting at x reaches y before it returns to x. So, as known, $P_{\text{esc}}(x, y) = \dfrac{\gamma_{xy}}{\delta(x)}$ and hence

$$P_{\text{esc}}(x, y) = \frac{n}{\delta(x)(\nu_x(y) + \nu_y(x))}.$$

Some known results about the effective resistances can be proved by using (6). For instance, the following one is originally due to R.M. Foster [9]:

$$\sum_{(x,y)\in E} r_{xy} = \frac{1}{2}\sum_x \sum_{y\sim x} r_{xy} = \frac{1}{n}\sum_x \sum_{y\sim x} \nu_x(y) = -\frac{1}{n}\sum_x \mathcal{L}\nu_x(x) = n - 1.$$

Although the effective resistance and the above concepts have been expressed here in terms of the solution of a Poisson Problem, they can also be obtained from the solution

of one of the Dirichlet Problems raised in (5). Namely,

$$\begin{aligned} \mathcal{L}v &= 0 \text{ in } V - \{x, y\} \\ v(x) &= 1 \\ v(y) &= 0. \end{aligned} \tag{7}$$

To see that, let u be any solution of the Poisson Problem $\mathcal{L}u = \varepsilon_x - \varepsilon_y$. Then, $v = \gamma_{xy}(u - u(y))$ verifies $v(x) = 1$ by definition of γ_{xy} and $\mathcal{L}v = \gamma_{xy}\mathcal{L}u = \gamma_{xy}(\varepsilon_x - \varepsilon_y) = 0$ in $V - \{x, y\}$, and hence v is the solution of (7). On the other hand, $I(v) = \gamma_{xy}^2 I(u) = \gamma_{xy}$, since $I(u) = r_{xy}$. If we take again $u = G_x^y$ we get that

$$v = \frac{\nu_y - \nu_x + \nu_x(y)}{\nu_x(y) + \nu_y(x)}.$$

Alternatively, from Proposition 3.3, v can be expressed by

$$v = \frac{\nu_y - \nu_{xy}}{\nu_y(x)},$$

where ν_{xy} is the equilibrium measure for $V - \{x, y\}$.

In the context of random walks, $v(z)$ is the probability that if the walk starts at z it will reach x before it reaches y.

Let us point out that to compute the effective resistance between any pair of vertices it suffices to solve n equilibrium problems and hence n Linear Programming Problems. However, it is clear that the number of problems that we have to solve, could be drasticly reduced if we have additional information about the graph structure. The most striking case appears when Γ is a distance-regular graph in which it suffices to solve a unique Linear Programming Problem.

This kind of graphs have been studied by N. Biggs [3] and the authors [1]. Specifically, we showed that in a distance-regular graph for each vertex x the equilibrium measure ν_x is distributed by distances. This means that there exist $0 < q_1 < \cdots < q_d$ such that $\nu_x(y) = q_i$ iff $d(x, y) = i$. Therefore, in a distance-regular graph for any pair of vertices x, y, the effective resistance and the escape probability between them are:

$$r_{xy} = 2\frac{q_i}{n} \quad \text{and} \quad P_{\text{esc}}(x, y) = \frac{n}{2kq_i} \quad \text{if} \quad d(x, y) = i,$$

where k is the degree of Γ. In particular, $q_1 = (n-1)/k$, since $1 - n = \mathcal{L}\nu_x(x) = -\sum_{z \sim x} \nu_x(z) = -kq_1$. Hence, the effective resistance between adjacent vertices is $r_{xy} = (n-1)/m$, a well known result due also to Foster.

We must note that there exist non distance-regular graphs verifying that the equilibrium measures ν_x are distributed by distances. Therefore, the above results hold in such

graphs. For instance, this is the case of the "Buckyball" graph (see [5] for the definition of such graph.)

Finally, we present the equilibrium measures and the effective resistances of some very simple graphs for which the effective resistance is well-known.

If Γ is a cycle, C_n, then $\nu_x(y) = \dfrac{d(x,y)(n-d(x,y))}{2}$ and $r_{xy} = \dfrac{d(x,y)(n-d(x,y))}{n}$.

If Γ is a path, $P_n = \{x_1, \ldots, x_n\}$, then

$$\nu_{x_i}(x_j) = \frac{1}{2} \begin{cases} d(x_i,x_j)(d(x_i,x_n) + d(x_j,x_n) + 1) & \text{if } \ 1 \le i < j \le n \\ d(x_i,x_j)(d(x_i,x_1) + d(x_j,x_1) + 1) & \text{if } \ 1 \le j < i \le n \end{cases}$$

and $r_{x_i x_j} = d(x_i, x_j)$.

If Γ is a complete graph, K_n, then $\nu_x(y) = 1$ and $r_{xy} = 2/n$.

If Γ is a bipartite complete graph, $K_{p,q}$, with partite sets V_0 and V_1, then

$$\nu_x(y) = \begin{cases} \dfrac{n}{q} & x, y \in V_0 \\ \dfrac{n-1}{q} & x \in V_0, y \in V_1 \\ \dfrac{n-1}{p} & x \in V_1, y \in V_0 \\ \dfrac{n}{p} & x, y \in V_1 \end{cases} \quad \text{and} \quad r_{xy} = \begin{cases} \dfrac{2}{q} & x, y \in V_0 \\ \dfrac{n-1}{m} & x \in V_0, y \in V_1 \\ \dfrac{2}{p} & x, y \in V_1. \end{cases}$$

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References

[1] E. Bendito, A. Carmona and A. M. Encinas, Shortest Paths in Distance-Regular Graphs, *Europ. J. Combinatorics* **21** (2000), 153-166.

[2] A. Beurling and J. Deny, Espaces de Dirichlet, *Acta Math.* **99** (1958), 203–224.

[3] N.L. Biggs, Potential theory on distance-regular graphs, *Combin., Probab. Comput.* **2** (1993), 243–255.

[4] N.L. Biggs, Algebraic potential theory on graphs, *Bull. London Math. Soc.* **29** (1997), 641–682.

[5] F.R.K. Chung,"Spectral Graph Theory", CBMS 92, American Mathematical Society, 1997.

[6] F.R.K. Chung and S.T. Yau, Discrete Green's functions, *J. Comb. Theory (A)* **91** (2000), 191–214.

[7] P.G. Ciarlet,"Introduction a L'Analyse Numérique Matricielle et à L'Optimisation," Masson, 1988.

[8] P.G. Doyle and J.L. Snell, "Random Walks and Electric Networks," Math. Assoc. America, 1984.

[9] R.M. Foster, The average impedance of an electrical network. In *Contributions to Applied Mechanics,* J.W. Edwards, Ann Arbor, Michigan (1949), 333–340.

[10] T. Kayano and M. Yamasaki, Dirichlet finite solution of Poisson equations on a infinite network, *Hiroshima Math. J.* **12** (1982), 569–579.

[11] T. Kayano and M. Yamasaki, Boundary limit of discrete Dirichlet potentials, *Hiroshima Math. J.* **14** (1984), 401–406.